Chemical Physics Cumulative Exam
September 23, 2002

This exam consists of 9 questions worth a total of 100 points, The point value for each question is indicated below. Show ALL work. Insufficient justifications will result in a loss of points.

Consider the one-dimensional harmonic oscillator with the hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_c^2\hat{x}^2$$

described using occupation number representation

$$\hat{x} = (\frac{\hbar}{2\omega_c m})^{\frac{1}{2}}(a + a^\dagger)$$

$$\hat{p} = -i(\frac{\hbar\omega_c m}{2})^{\frac{1}{2}}(a - a^\dagger)$$

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle$$

$$a^\dagger|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$

$$N|n\rangle = n|n\rangle$$

where

$\hat{x}$ is the coordinate operator,
$\hat{p}$ is the momentum operator,
a is the annihilation operator,
$a^\dagger$ is the creation operator,
$N = a^\dagger a$ is the number operator,
ω_c is the angular frequency of the corresponding classical harmonic oscillator, and
m is the mass of the oscillating system.

1. (20 pts) Calculate the uncertainties Δx and Δp and prove $\Delta x \Delta p = (n+\frac{1}{2})\hbar$ for $|n\rangle$, the n-th eigenstate of the quantum mechanical oscillilator.

2. (10 pts) From the commutator relation $[a, a^\dagger] = 1$, prove $a(a^\dagger)^n = (a^\dagger)^n a + n(a^\dagger)^{(n-1)}$.

3. (10 pts) Verity the commutator relation $[a, a^\dagger] = 1$ by operating with the commutator on $|n\rangle$ and using the relations for $a|n\rangle$ and $a^\dagger|n\rangle$ given above to evaluate the result.

4. (10 pts) Starting from the hamiltonian shown above, show that $E = \hbar\omega_c/2$ for $|0\rangle$, the ground state of the one-dimensional harmonic oscillator.

5. (5 pts) Since $|0\rangle$ is the lowest energy eigenstate, $a|0\rangle$ must be zero. Provide a short justification for this conclusion.

6. (10 pts) The annihilation operator can also be expressed in terms of the dimensionless quantity $y = (\sqrt{m\omega_c/\hbar})\hat{x}$:

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)$$

Using the relations for $\hat{x}$ and $\hat{p}$ given above, derive this expression.

7. (10 pts) Use the expression for a in terms of y in Question 6 and the fact that $a|0\rangle = 0$ as stated in Question 5 to derive the ground state wavefunction for the one-dimensional harmonic oscillator in terms of y.

8. (10 pts) Calculate the expectation value of the kinetic energy for $|n\rangle$.

9. (15 pts) Evaluate $\langle n|\hat{x}^3|n+1\rangle$.