1 $(\Delta X)^2 = \langle X^2 \rangle - \langle x \rangle^2$

$$(\Delta X)^2 = \langle n | \frac{\hbar}{2\omega_c m}(a+a^\dagger)^2 | n \rangle - \left(\langle n | \left(\frac{\hbar}{2\omega_c m}\right)^{1/2} (a+a^\dagger) | n \rangle \right)^2$$

$$= \frac{\hbar}{2\omega_c m} \langle n (a+a^\dagger)^2 | n \rangle - \frac{\hbar}{2\omega_c m}\left(\langle n | (a+a^\dagger) | n \rangle \right)^2$$

$$= \frac{\hbar}{2\omega_c m}\Big[\langle n | (\cancel{a^2} + a^\dagger a + a a^\dagger + \cancel{a^{\dagger 2}}) | n \rangle - \langle n | (a+a^\dagger) | n \rangle \Big]^2$$

$$= \frac{\hbar}{2\omega_c m}\Big[\langle n | a^\dagger a + a a^\dagger | n \rangle \Big\} - \Big[\langle n | (a+a^\dagger | n \rangle \Big]^2$$

$$= \frac{\hbar}{2\omega_c m}\Big[\langle n | a^\dagger a | n \rangle + \langle n | a a^\dagger | n \rangle - \Big\{ \langle n | a | n \rangle + \langle n | a^\dagger | n \rangle \Big] \Big]^2$$

$$\Big[\langle n | N | n \rangle + \langle n | a (n+1)^{1/2} | n+1 \rangle - \Big\{ \langle n | n^{1/2} | n-1 \rangle + \langle n | (n+1)^{1/2} | n+1 \rangle \Big]^2$$

$$\frac{\hbar}{2\omega_c m}\Big[\langle n | n | n \rangle + \langle n | (n+1)^{1/2} a | n+1 \rangle - \Big\{ \langle n | n^{1/2} | n-1 \rangle + \langle n (n+1)^{1/2} | n+1 \rangle \Big]^2$$

$$\frac{\hbar}{2\omega_c m}\Big[n + \langle n | (n+1)^{1/2} (n+1)^{1/2} | n \rangle - \Big[\langle n | n^{1/2} | n-1 \rangle + \langle n (n+1)^{1/2} | n+1 \rangle \Big]^2$$

$$(\Delta X)^2 \frac{\hbar}{2\omega_c m}\Big[n + n+1 - \phi - \phi \Big] = \boxed{\frac{\hbar}{2\omega_c m}\big[2n+1\big]}$$

$$\Delta X = \left(\frac{\hbar}{2\omega_c m}\right)^{1/2} \big[2n+1\big]^{1/2}$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$$

$$= \langle n| -\frac{\hbar\omega_c m}{2}(a-a^+)^2|n\rangle - \left[\langle n|-i\left(\frac{\hbar\omega_c m}{2}\right)^{1/2}(a-a^+)|n\rangle\right]^2$$

$$= -\frac{\hbar\omega_c m}{2}\left[\langle n|(a-a^+)^2|n\rangle\right] - \left(-\frac{\hbar\omega_c m}{2}\right)\left[\langle n|(a-a^+)|n\rangle\right]^2$$

$$= -\frac{\hbar\omega_c m}{2}\left[\langle n|\cancel{a^2}^{\,\phi}-a^+a-aa^++\cancel{a^{+2}}^{\,\phi}|n\rangle\right] - \left[\langle n|(a-a^+)|n\rangle\right]^2$$

$$= -\frac{\hbar\omega_c m}{2}\left[-\langle n|a^+a|n\rangle - \langle n|aa^+|n\rangle\right] - \left[\langle n|a|n\rangle - \langle n|a^+|n\rangle\right]^2$$

$$= -\frac{\hbar\omega_c m}{2}\left[-n - \langle n|(n+1)^{1/2}a|n+1\rangle\right] - \left[\langle n|n^{1/2}|n-1\rangle - \langle n|(n+1)^{1/2}|n+1\rangle\right]^2$$

$$= -\frac{\hbar\omega_c m}{2}\left[-n - \langle n|(n+1)^{1/2}(n+1)^{1/2}|n\rangle\right] - \phi$$

$$(\Delta p)^2 = -\frac{\hbar\omega_c m}{2}\left[-n-(n+1)\right] = \frac{\hbar\omega_c m}{2}\left[2n+1\right]$$

$$\Delta p = \left(\frac{\hbar\omega_c m}{2}\right)^{1/2}\left[2n+1\right]^{1/2}$$

$$\Delta X\Delta p = \left(\frac{\hbar}{2\cancel{\omega_c}\cancel{m}}\right)^{1/2}\left(\frac{\hbar\cancel{\omega_c}\cancel{m}}{2}\right)^{1/2}(2n+1)^{1/2}(2n+1)^{1/2}$$

$$= \frac{\hbar}{2}(2n+1) = \boxed{\hbar\left(n+\frac{1}{2}\right)}$$ ok

2. $aa^{\dagger n} = aa^{\dagger}a^{\dagger(n-1)} = (1+a^{\dagger}a)a^{\dagger(n-1)}$

$= a^{\dagger(n-1)} + a^{\dagger}aa^{\dagger(n-1)}$

$= a^{\dagger(n-1)} + a^{\dagger}(aa^{\dagger})a^{\dagger(n-2)} = a^{\dagger(n-1)} + a^{\dagger}(1+a^{\dagger}a)a^{\dagger(n-2)}$

$= a^{\dagger(n-1)} + a^{\dagger}\left(a^{\dagger(n-2)}\right) + a^{\dagger 2}aa^{\dagger(n-2)}$

$= a^{\dagger(n-1)} + a^{\dagger(n-1)} + a^{\dagger 2}aa^{\dagger(n-2)}$

$= 2a^{\dagger(n-1)} + a^{\dagger 2}aa^{\dagger(n-2)}$ until n

$\boxed{= na^{\dagger(n-1)} + a^{\dagger n}a}$

3. $(aa^{\dagger} - a^{\dagger}a)|n\rangle = aa^{\dagger}|n\rangle - a^{\dagger}a|n\rangle$

$= a(n+1)^{1/2}|n+1\rangle - a^{\dagger}(n)^{1/2}|n-1\rangle$

$= (n+1)^{1/2}a|n+1\rangle - n^{1/2}a^{\dagger}|n-1\rangle$

$= (n+1)^{1/2}(n+1)^{1/2}|n\rangle - n^{1/2}n^{1/2}|n\rangle$

$= n+1|n\rangle + n|n\rangle$

$= n+1-n = \boxed{1}$

4. $\frac{p^2}{2m} + \frac{1}{2}m\omega_c^2\hat{x}^2|n\rangle = E|n\rangle$

$-\frac{\left(\frac{\hbar\omega_c m}{2}\right)(a-a^{\dagger})^2}{2m} + \frac{1}{2}m\omega_c^2\left(\frac{\hbar}{2\omega_c m}\right)(a+a^{\dagger})^2|n\rangle = E|n\rangle$

$= -\frac{\hbar\omega_c}{4}(a-a^{\dagger})^2 + \frac{\hbar\omega_c}{4}(a+a^{\dagger})^2|n\rangle = E|n\rangle$

$= -\frac{\hbar\omega_c}{4}\left\{\left[(a-a^{\dagger})^2 - (a+a^{\dagger})^2\right]|n\rangle\right\} = E|n\rangle$

$= -\frac{\hbar\omega_c}{4}\left[\cancel{a^2} - aa^{\dagger} - a^{\dagger}a + \cancel{a^{\dagger 2}} - \cancel{a^2} - aa^{\dagger} - a^{\dagger}a - \cancel{a^{\dagger 2}}\right]|n\rangle = E|n\rangle$

$= +\frac{\hbar\omega_c}{\cancel{4}\,2}\left[+\cancel{2}aa^{\dagger} + \cancel{2}a^{\dagger}a\right]|n\rangle = E|n\rangle$

$= \frac{\hbar\omega_c}{2}\left[aa^{\dagger} + a^{\dagger}a\right]|n\rangle = E|n\rangle$

$$\frac{\hbar\omega_c}{2}\left[aa^+ + a^+a\right]|n\rangle = E|n\rangle$$

$$\frac{\hbar\omega_c}{2}\left[aa^+|n\rangle + a^+a|n\rangle\right] = E|n\rangle$$

$$\frac{\hbar\omega_c}{2}\left[a(n+1)^{1/2}|n+1\rangle + \cancel{n}|n\rangle\right] = E|n\rangle$$

$$\frac{\hbar\omega_c}{2}\left[(n+1)^{1/2}a|n+1\rangle + n|n\rangle\right] = E|n\rangle$$

$$\frac{\hbar\omega_c}{2}\left((n+1)^{1/2}(n+1)^{1/2}|n\rangle + n|n\rangle\right] = E|n\rangle$$

$$\frac{\hbar\omega_c}{2}\left[n+1+n\right]|n\rangle = E|n\rangle$$

$$\frac{\hbar\omega_c}{2}(2n+1) = E_n \qquad \text{for } n=0$$

$$\boxed{\frac{\hbar\omega_c}{2} = E_0}$$

5. $a|0\rangle$ must be zero since

$$a|n\rangle = n^{1/2}|n-1\rangle \quad \text{if } n=0, \text{ then}$$

$n=1$ does not exist, and $a|0\rangle = 0$ must be true

7. $$a|0\rangle = 0 \quad ; \quad a\Psi_0 = 0$$

$$\cancel{a\Psi_0}\ \frac{1}{\cancel{\sqrt{2}}}\left(y + \frac{d}{dy}\right)\Psi_0 = 0$$

$$\left(y + \frac{d}{dy}\right)\Psi_0 = 0 \qquad y\Psi_0 + \frac{d\Psi_0}{dy} = 0$$

$$\frac{d\Psi_0}{dy} = -y\Psi_0$$

5

$$\int \frac{d\psi_0}{\psi_0} = \int -y\,dy$$

$$\ln \psi_0 = -\frac{y^2}{2} \qquad \boxed{\psi_0 = C_0 e^{-y^2/2}}$$

8. $$\langle n | \frac{p^2}{2m} | n \rangle = \langle n | \frac{\left(-i\left(\frac{\hbar\omega_c m}{2}\right)^{1/2}\right)^2 (a-a^\dagger)^2}{2m} | n \rangle$$

$$= \langle n | -\frac{\hbar\omega_c \cancel{m}}{4\cancel{m}} (a-a^\dagger)^2 | n \rangle$$

$$= -\frac{\hbar\omega_c}{4} \langle n | (a-a^\dagger)^2 | n \rangle$$

$$= -\frac{\hbar\omega_c}{4} \langle n | \overset{0}{\cancel{a^2}} - aa^\dagger - a^\dagger a + \overset{0}{\cancel{a^{\dagger 2}}} | n \rangle$$

$$= +\frac{\hbar\omega_c}{4} \langle n | + aa^\dagger + a^\dagger a | n \rangle$$

$$= \frac{\hbar\omega_c}{4} \left(\langle n | a a^\dagger | n \rangle + \langle n | a^\dagger a | n \rangle \right)$$

$$= \frac{\hbar\omega_c}{4} \left[\langle n | a (n+1)^{1/2} | n+1 \rangle + n \right]$$

$$= \frac{\hbar\omega_c}{4} \left[\langle n | (n+1)^{1/2} (n+1)^{1/2} | n \rangle + n \right]$$

$$= \frac{\hbar\omega_c}{4} \left[n + 1 + n \right] = \frac{\hbar\omega_c}{4} (2n+1) = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega_c}$$

9. $\langle n|\hat{x}^3|n+1\rangle \qquad \hat{x}^3 = \left(\frac{\hbar}{2\omega_c m}\right)^{3/2}(a+a^\dagger)^3$

$$\langle n|\left(\frac{\hbar}{2\omega_c m}\right)^{3/2}(a+a^\dagger)^3|n+1\rangle$$

$$\left(\frac{\hbar}{2\omega_c m}\right)^{3/2}\left[\langle n|(a+a^\dagger)^3|n+1\rangle\right] = \frac{\hbar^{3/2}}{2\omega_c m}\left[\langle n|(a+a^\dagger)(a^2+aa^\dagger+a^\dagger a+a^{\dagger 2})|n+1\rangle\right]$$

$$\begin{array}{r} a^2+aa^\dagger+a^\dagger a+a^{\dagger 2} \\ a+a^\dagger \\ \hline a^3+a^2a^\dagger+aa^\dagger a+aa^{\dagger 2}+a^\dagger a^2+a^\dagger a a^\dagger+a^{\dagger 2}a+a^{\dagger 3} \end{array}$$

$$\frac{\hbar^{3/2}}{2\omega_c m}\left[\langle n|\cancelto{0}{a^3}+a^2a^\dagger+aa^\dagger a+\cancelto{0}{aa^{\dagger 2}}+a^\dagger a^2+\cancelto{0}{a^\dagger a a^\dagger}+\cancelto{0}{a^{\dagger 2}a}+\cancelto{0}{a^{\dagger 3}}|n+1\rangle\right]$$

require lower by 1 to have non zero term.

$$\frac{\hbar^{3/2}}{2\omega_c m}\left[\langle n|\,a^2a^\dagger+aa^\dagger a+a^\dagger a^2\,|n+1\rangle\right]$$

$$\frac{\hbar^{3/2}}{2\omega_c m}\left[\langle n|a^2a^\dagger|n+1\rangle+\langle n|aa^\dagger a|n+1\rangle+\langle n|a^\dagger a^2|n+1\rangle\right]$$

$$\frac{\hbar^{3/2}}{2\omega_c m}\left[\langle n|a^2(n+2)^{1/2}|n+2\rangle+\langle n|aa^\dagger(n+1)^{1/2}|n\rangle+\langle n|a^\dagger a(n+1)^{1/2}|n\rangle\right]$$

$$\left[\langle n|a(n+2)^{1/2}(n+2)^{1/2}|n+1\rangle+\langle n|a(n+1)^{1/2}(n+1)^{1/2}|n+1\rangle+\langle n|a^\dagger(n+1)^{1/2}n^{1/2}|n-1\rangle\right]$$

$$\left[\langle n|(n+2)(n+1)^{1/2}|n\rangle+\langle n|(n+1)(n+1)^{1/2}|n\rangle+\langle n|n(n+1)^{1/2}|n\rangle\right.$$

$$\frac{\hbar^{3/2}}{2\omega_c m}\left[\cancel{(n+1)^{a}+(n+1)^2+n(n+1)^{1/2}}\right] = \boxed{\frac{\hbar^{3/2}}{2\omega_c m}(n+1)^{1/2}[3n+3]}$$

$$\left[(n+2)(n+1)^{1/2}+(n+1)(n+1)^{1/2}+n(n+1)^{1/2}\right]$$

9. $H = \varepsilon_1 a^\dagger a + \varepsilon_2 (a + a^\dagger)$

$\varepsilon_1 a^\dagger a + \varepsilon_2 (a + a^\dagger) |n\rangle = E |n\rangle$

$\varepsilon_1 a^\dagger a |n\rangle + \varepsilon_2 (a + a^\dagger) |n\rangle = E |n\rangle$

$\varepsilon_1 N |n\rangle + \varepsilon_2 a |n\rangle + \varepsilon_2 a^\dagger |n\rangle = E |n\rangle$

$+ \varepsilon_2 (n)^{1/2} |n-1\rangle + \varepsilon$

6. $\hat{x} = \left(\frac{\hbar}{2\omega_c m}\right)^{1/2} (a + a^\dagger)$

$\hat{x} \left(\frac{2\omega_c m}{\hbar}\right)^{1/2} = a + a^\dagger \qquad y = \left(\frac{m\omega_c}{\hbar}\right)^{1/2} \hat{x}$

$\sqrt{2}\, y = a + a^\dagger \qquad a = \sqrt{2}\, y - a^\dagger$

$\hat{p} = -i \left(\frac{\hbar \omega_c m}{2}\right)^{1/2} (a - a^\dagger) \qquad \hat{p} = \frac{\hbar}{i} \frac{d}{d\hat{x}}$

$\hat{p} = +\frac{\hbar}{i} \frac{d}{d\hat{x}} = \mp i \left(\frac{\hbar \omega_c m}{2}\right)^{1/2} (a - a^\dagger) \qquad dy = \left(\frac{m\omega_c}{\hbar}\right)^{1/2} d\hat{x}$

$\frac{\hbar}{i} \frac{d}{dx} \mp i \left(\frac{\hbar \omega_c m}{2}\right)^{1/2} (a - a^\dagger)$

$-\frac{\hbar}{i^2} \left(\frac{2}{\hbar \omega_c m}\right)^{1/2} \frac{d}{dx} = a - a^\dagger$

$+1 \left(\frac{\hbar^2 2}{\hbar \omega_c m}\right)^{1/2} \frac{d}{dx} = a - a^\dagger, \quad -\left(\frac{2\hbar}{\omega_c m}\right)^{1/2} \frac{d}{dx} = a - a^\dagger$

$+\left(\frac{2 \cancel{\hbar}}{\cancel{\omega_c m}}\right)^{1/2} \frac{d}{dy \left(\frac{\hbar}{m\omega_c}\right)^{1/2}} = a - a^\dagger$

$+\sqrt{2} \frac{d}{dy} = a - a^\dagger \qquad a^\dagger = a + \sqrt{2} \frac{d}{dy}$

6 from $\hat{x}$ $a = \sqrt{2}\,y - a^{+}$

from $\hat{p}$ $a^{+} = a + \sqrt{2}\,\frac{d}{dy}$

$$a = \sqrt{2}\,y - \left(a + \sqrt{2}\,\frac{d}{dy}\right)$$

$$= \sqrt{2}\,y - a + \sqrt{2}\,\frac{d}{dy}$$

$$2a = \sqrt{2}\left(y + \frac{d}{dy}\right)$$

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)$$